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OPTIMIZATION OF THE BOUNDARY REGIME IN THE PROBLEM OF THE TRANSFER AND ABSORPTION OF MATERIAL IN A POROUS MEDIUM*

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Problems of optimizing the mode of supplying a materail to the surface of a porous medium are considered for the case when the material is carried by a flow passing through the medium, and is partly absorbed by it. Such a problem arises e.g. when fertilizers are placed on the surface of soil, or when chemical components are supplied to the boundary of a medium within which a reaction with absorption takes place.

Let us consider, to be specific, the problem of the transport of fertilizers in soil. Their transport will be governed by the flow of moisture. The dynamic behaviour of the moisture in a layer of soil $0 \leqslant z \leqslant l$ is described by the problem

$$u_t + q_z = -F(z, t); \quad q|_{z=0} = q_0(t), \ R(q, u)|_{z=l} = 0, \ u|_{t=0} = \varphi(z) \tag{1}$$

Here t is the time, z is the vertical coordinate ($z = 0$ at the surface of the medium), u is the moisture density, q is its flux, $q_0(t)$ is a given function determined by the precipitation, watering and evaporation from the soil surface, R is an operator describing the boundary conditions when $z = l$. $F(z, t)$ is the amount of moisture demanded by the plant roots.

Let a layer of material A, in crystalline form, be present at the soil surface at the time $t > 0$. We denote the concentration of dissolved material in water passed through this layer by $\chi(t)$. In the linear approximation we have

$$\chi H(q_0) = \beta(c_0 - \chi); \quad H(x) = \begin{cases} x, & x \geqslant 0 \\ 0, & x < 0 \end{cases} \tag{2}$$

where $\beta(t)$ is the exchange coefficient between the solid and liqud phase of A, depending on the amount of material present at the surface in crystalline form, and $C_0 = \text{const}$ is the saturation concentration of A.

The dynamics of the material transported by the moisture is described by the problem

$$L(c, u) = 0; \ \chi(t) H(q_0(t)) = R_1(c, q_0)|_{z=0} \tag{3}$$

$$R_2(c, q)|_{z=l} = 0, \quad c|_{t=0} = 0$$

where c is the concentration of the material in water, L is a differential operator describing the transport, diffusion and sorption of the material A depending on the flow q and the moisture density u. R_1 and R_2 determine the boundary conditions. For example, $R_1(c, q) = qc - a(q)c_z$, $R_2(c, q) = c_z$ where $a(q)$ is the diffusion coefficient.

Since the experimental data are not very accurate due to the inhomogeneity of the soil, it is usually sufficient to consider the models to be linear in c /1/. We shall assume L, R_1 and R_2 to be linear in c, and that a solution of problem (1), (3) exists, is unique, non-negative at any $\chi \geqslant 0$, and that the integral $\int_0^l c(z, t)\,dz$ is bounded, provided that the amount of material passed into the medium from the surface

$$\int_0^t \chi H(q_0)\,d\tau$$

is finite. The relations (1)–(3) constitute a model of the process.

Let us assume that the surface can be covered by a layer of crystalline material A of differing density, i.e. that $\beta(t)$ varies within certain limits $0 \leqslant \beta(t) \leqslant \beta_m = \text{const}$. Let the plants require the material and water, and let the function F be bounded. Then the amount of material required over the time T will be

$$Q = \int_0^T dt \int_0^l F(z, t)\, c(z, t)\, dz$$

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At the same time, the consumption of the material at the surface will be

$$P = \int_0^T \chi(t) H(q(t))\, dt$$

Let us formulate the following optimization problem. It is required to determine the control function $\beta(t)$, varying within the limits shown, for which the demand Q is equal to the given value Q_0, and the consumption P is minimal. We shall assume here that the quantity Q_0 satisfies the inequality $0 < Q_0 < Q_m$ where Q_m is the value of Q at $\beta = \beta_m$.

Let us denote by $G(z, t, \tau)$ the solution of the problem (1), (3) with boundary condition $R_1(c, t) = \delta(t - \tau)$. Problem (3) is linear in c, and therefore for any χ we have

$$c(z, t) = \int_0^t \chi(\tau) H(q_0(\tau)) G(z, t, \tau)\, d\tau$$

Substituting this relation into the expression for Q and changing the order of integration, we obtain

$$Q = \int_0^T \chi(\tau) H(q_0(\tau)) B(\tau)\, d\tau; \quad B(\tau) = \int_0^l dz \int_0^T F(z, t) G(z, t, \tau)\, dt \tag{4}$$

According to the condition

$$0 \leqslant F < \infty, \quad G \geqslant 0, \quad \int_0^l G\, dz < \infty$$

therefore $0 \leqslant B < \infty$. Consider the functional

$$I = P\lambda - Q = \int_0^T \chi(\tau) H(q_0(\tau)) [\lambda - B(\tau)]\, d\tau \tag{5}$$

where λ is an arbitrary fixed number. For a given value of Q, the minimum of P and I is attained on the same curves. Let us denote by T_0 the set of t for which $q_0(t) = 0$; $T^+(\lambda)$ the set of t for which $q_0(t) > 0$ and $B(t) > \lambda$; $T^-(\lambda)$ the set of t for which $q_0(t) > 0$ and $B(t) < \lambda$.

The value of χ on the set T_0 does not affect (4) and (5). When $q_0 > 0$, we have from (2) $\chi = \beta c_0 (\beta + q_0)^{-1}$, which is a monotonically increasing function of β. It is clear that the value of (5) will be minimal at any λ, provided that $\chi = 0$ on $T^-(\lambda)$ and $\chi = \chi_m(t) = c_0\beta_m(\beta_m + q_0(t))^{-1}$ when $t \in T^+(\lambda)$. We choose, respectively, $\beta = 0$ on the set $T^-(\lambda)$ and $\beta = \beta_m$ or $T^+(\lambda)$, and here we have $Q = f(\lambda) = \int_{T^+(\lambda)} \chi_m q_0 B\, d\tau$.

When λ increases, the set $T^+(\lambda)$ does not grow larger, therefore $f(\lambda)$ is a monotonically non-increasing function. Generally speaking, $f(\lambda)$ will not be a continuous function here.

When $\lambda > 0$, $T^-(\lambda) = 0$ and $f(\lambda) = \int_0^T \chi_m q_0 B\, d\tau > Q_0$ in accordance with the condition. Since B is bounded when $\lambda \to \infty$, the set $T^+(\lambda) = 0$ and $f(\lambda) = 0 < Q_0$. Therefore a number λ_0 can be found such, that either $f(\lambda_0) = Q_0$ or $f(\lambda_0) > Q_0$, $f(\lambda_0 - 0) < Q_0$, the function $f(\lambda)$ has a discontinuity at the point λ_0 and $B(t) = \lambda_0$ on the finite set $T_{\lambda_0} = T^+(\lambda_0) - T^+(\lambda_0 + 0)$. In the first case the condition $Q = Q_0$ holds for $\beta(t) = \beta_m$ when $t \in T^+(\lambda_0)$ and $\beta_{(t)} = 0$ when $t \in T^-(\lambda_0)$, and a strong minimum of the functional (4) is attained in the class of admissible variations in β. The second case differs in the fact that the function β on the set T_{λ_0} must satisfy the relation

$$\int_{T_{\lambda_0}} \frac{\beta c_0 q_0}{\beta + q_0}\, d\tau = Q_0 - \int_{T^+(\lambda_0+0)} \frac{\beta_m c_0 q_0}{\beta_m + q_0}\, d\tau \tag{6}$$

As a result, we have the following theorem.

Theorem. A solution of the optimization problem in question exists, and is represented by the function $\beta(t)$ of admissible class, furnishing a minimum to the functional P at fixed $Q = Q_0$. The function is determined as follows: $\beta(t) = \beta_m$ for $t \in T^-(\lambda_0) = \{t : q_0(t) > 0,\ B(t) < \lambda_0\}$; $\beta(t) = 0$ for $t \in T^-(\lambda_0) = \{t : q_0(t) > 0, B(t) > \lambda_0\}$; $\beta(t)$ is arbitrary for $t \in T_0 = \{t : q_0(t) = 0\}$; $\beta(t)$ and satisfies relation (6), otherwise it is arbitrary for $t \in T_{\lambda_0} = \{t : q_0(t) > 0, B(t) = \lambda_0\}$; λ_0 is a number (it exists and is real) such, that

$$\int_{T^+(\lambda_0)} \frac{\beta_m c_0 q_0}{q_0 - \beta_m}\, d\tau \leqslant Q_0 \leqslant \int_{T^+(\lambda_0)+T_{\lambda_0}} \frac{\beta_m c_0 q_0}{\beta_m - q_0}\, d\tau$$

The solution of the problem is unique on the set

$$\{t : q_0(t) > 0,\ B(t) = \lambda_0\}$$

To find the set $T^+(\lambda)$, we must determine the function $B(\tau)$. It can be done, in general, numerically. It often happens however, when the experimental results show a considerable spread caused by the spatial inhomogeneity of the medium, that relatively simple models are found to offer better agreement with the experimental results, and we can obtain an analytic solution for them.

Let us consider the following case. We shall be describing the dynamics of the moisture and salt fluxes averaged over several days. Let $q = b(u - u_0)$ where b and u_0 are constants and $q_0(t)$ exceeds the total demand for the moisture by the roots, i.e. $q > 0$ everywhere. In accordance with (2) we put $L(c, u) = mc_t + qc_z - ac_{zz}$ where $a = a_0 q$ is the coefficient of diffusion of the material A, a_0 and m are constants and $R_1 = qc - ac_z$. Considering, for simplicity, the problem on the segment $0 < z < \infty$, $t > -\infty$, we obtain

$$bq_t + q_z = -F(z, t), \quad q|_{z=0} = q_0(t) \tag{7}$$
$$mc_t + qc_z = a_0 q c_{zz}, \quad c|_{t<0} = \chi|_{t<0} = 0$$
$$(c - a_0 c_z)|_{z=0} = \chi(t) = \frac{\beta(t) c_0}{\beta(t) + q_0(t)}$$

We solve the problem for q and introduce the function

$$v(z, t, \tau) = \int_\tau^t q(z, \theta)\, d\theta = \int_\tau^t \left[q_0(\theta - bz) - \int_0^z F(x, \theta - bx)\, dx \right] d\theta \tag{8}$$

We regard v and z as new variables, and τ as a parameter. Then the equation for c in (7) will be transformed into an equation with constant coefficients. In the present case $G(z, t, \tau) = K(z, v(z, t, \tau))$ where $K(z, x)$ satisfies the relation

$$mK_x + K_z = a_0 K_{zz}; \quad x > -\infty, \quad 0 < z < \infty$$
$$K|_{x<0} = 0, \quad (K - a_0 K_z)|_{z=0} = \delta(x)$$

Solving this problem we find G, and obtain the following expression for B:

$$B(\tau) = \int_0^\infty dz \int_\tau^T F(z, t) \left[\frac{\exp(-\xi^2)}{\sqrt{\pi a_0 m v}} - \frac{\exp(z/a_0)\operatorname{erfc}(\xi)}{2a_0 m} \right] dt, \quad \xi = \frac{zm + v}{a_0 m v} \tag{9}$$

Thus the relation $B(\tau)$ determined by the form of the function $F, q(u)$ and of the operators L, R, R_1, R_2, is obtained in the present case in analytic form. To find the optimal conditions of supply of the material, we must have specific expressions for the functions $q_0(t)$ and $F(z, t)$. Knowing q_0 and F, we find v from (8), and then B from (9). Further, according to what was said above, we find λ_0 and the optimal mode for $\beta(t)$.

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